THERE ARE ENOUGH AZUMAYA ALGEBRAS ON SURFACES

STEFAN SCHRÖER

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Abstract. Using Maruyama's theory of elementary transformations, I show that the Brauer group surjects onto the cohomological Brauer group for separated geometrically normal algebraic surfaces. As an application, I infer the existence of nonfree vector bundles on proper normal algebraic surfaces.

Introduction

Generalizing the classical theory of central simple algebras over fields, Grothendieck [12] introduced the Brauer group $\mathrm{Br}(X)$ and the cohomological Brauer group $\mathrm{Br}'(X)$ for schemes.

Let me recall the definitions. The *Brauer group* $\mathrm{Br}(X)$ comprises equivalence classes of Azumaya algebras. Two Azumaya algebras $\mathcal{A}, \mathcal{B}$ are called equivalent if there are everywhere nonzero vector bundles $\mathcal{E}, \mathcal{F}$ with $\mathcal{A} \otimes \mathcal{E}nd(\mathcal{E}) \simeq \mathcal{B} \otimes \mathcal{E}nd(\mathcal{F})$. Let us define the *cohomological Brauer group* $\mathrm{Br}'(X)$ as the torsion part of the étale cohomology group $H^2(X, \mathbb{G}_m)$. Nonabelian cohomology gives an inclusion $\mathrm{Br}(X) \subset \mathrm{Br}'(X)$, and Grothendieck asked whether this is bijective.

It would be nice to know this for the following reason: The cohomological Brauer group is related to various other cohomology groups via exact sequences, and this is useful for computations. In contrast, it is almost impossible to calculate the Brauer group of a scheme directly from the definition. Here is a list of schemes with $\mathrm{Br}(X) = \mathrm{Br}'(X)$:

1. Schemes of dimension ≤ 1 and regular surfaces (Grothendieck [12]).
2. Abelian varieties (Hoobler [16]).
3. The union of two affine schemes with affine intersection (Gabber [6]).
4. Smooth toric varieties (DeMeyer and Ford [4]).

On the other hand, a nonseparated normal surface with $\mathrm{Br}(X) \neq \mathrm{Br}'(X)$ recently appeared in [5]. I wonder how the final answer to this puzzle will look like. The goal of this paper is to prove the following Theorem.

Theorem. *For separated geometrically normal algebraic surfaces, the inclusion* $\mathrm{Br}(X) \subset \mathrm{Br}'(X)$ *is a bijection.*

This adds some singular and nonprojective schemes to the preceding list. For quasiprojective surfaces, Hoobler ([17] Cor. 9) deduced the result directly from Gabber's Theorem on affine schemes. Without ample line bundles, a different approach is required. Indeed, my initial motivation was to disprove the Theorem, rather than to prove it. The new idea is to use Maruyama's theory of elementary transformations.

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Here is an application of the preceding result:

Theorem. *Each proper normal algebraic surface admits a nonfree vector bundle.*

It might easily happen that all line bundles are free [25]. The existence of nonfree vector bundles can be viewed as a generalization, in dimension two, of Winkelmann's Theorem [28], which asserts that each compact complex manifold has nonfree holomorphic vector bundles.

This paper has four sections. In the first section, I relate Azumaya algebras that are trivial on large open subsets to certain reflexive sheaves. In Section 2, we turn to normal surfaces and construct Azumaya algebras that are generically trivial by constructing the corresponding reflexive sheaves. This prepares the proof of the main Theorem, which appears in Section 3. The idea in the proof is to apply elementary transformations to Brauer–Severi schemes. The last section contains the existence result for nonfree vector bundles.

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1. Azumaya algebras via reflexive sheaves

In this section, we shall describe Azumaya algebras that have trivial Brauer class on certain large open subsets. Throughout, X will be a noetherian scheme. Let us call an open subset $U \subset X$ *thick* if it contains all points $x \in X$ with $\operatorname{depth}(\mathcal{O}_{X,x}) \leq 1$. In other words, $\operatorname{depth}_{X-U}(\mathcal{O}_X) \geq 2$. A coherent $\mathcal{O}_X$-module $\mathcal{F}$ is called *almost locally free* if it is locally free on some thick open subset $U \subset X$, and has $\operatorname{depth}_{X-U}(\mathcal{F}) \geq 2$. Such sheaves behave well under suitable restriction and extension functors:

Lemma 1.1. *Let $i : Y \subset X$ be a thick open subset. Then the restriction map $\mathcal{F} \mapsto i^*(\mathcal{F})$ and the direct image map $\mathcal{G} \mapsto i_*(\mathcal{G})$ induce inverse equivalences between the categories of almost locally free sheaves on X and Y, respectively.*

Proof. This is similar to the proof of [15] Theorem 1.12. Fix an almost locally free $\mathcal{O}_X$-module $\mathcal{F}$. First, we check that $\Gamma(V, \mathcal{F}) \to \Gamma(V \cap Y, \mathcal{F})$ is bijective for all affine open subsets $V \subset X$. Setting $A = V - V \cap Y$, we have an exact sequence of local cohomology groups

$$(1) \qquad 0 \longrightarrow H^0_A(V, \mathcal{F}) \longrightarrow H^0(V, \mathcal{F}) \longrightarrow H^0(V \cap Y, \mathcal{F}) \longrightarrow H^1_A(V, \mathcal{F}) \longrightarrow 0.$$

Since $\operatorname{depth}_A(\mathcal{F}) \geq 2$, the cohomology groups with supports vanish by [13] Theorem 3.8. Therefore, the map in the middle is bijective. As a consequence, the adjunction map $\mathcal{F} \to i_* i^*(\mathcal{F})$ is bijective, so that the restriction functor $\mathcal{F} \mapsto i^*(\mathcal{F})$ is fully faithful.

Second, we check that the functor $\mathcal{F} \mapsto i^*(\mathcal{F})$ is essentially surjective. Fix an almost locally free $\mathcal{O}_Y$-module $\mathcal{G}$. By [8] Corollary 6.9.8, the sheaf $\mathcal{G}$ extends to a coherent $\mathcal{O}_X$-module $\mathcal{M}$. I claim that $\mathcal{F} = \mathcal{M}^{\vee\vee}$ is almost locally free. This is

a local problem, so we may assume that there is a partial resolution $\mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{M}^\vee \to 0$ with coherent locally free sheaves, hence an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{L}_0^\vee \longrightarrow \mathcal{L}_1^\vee. \tag{2}$$

Let $A = X - U$, where $U \subset Y$ is a thick open subset on which $\mathcal{G}$ is locally free. The exact sequence (2) gives an inclusion $H^0_A(X, \mathcal{L}_0^\vee/\mathcal{F}) \subset H^0_A(X, \mathcal{L}_1^\vee)$ and an exact sequence of local cohomology groups

$$H^0_A(X, \mathcal{L}_0^\vee/\mathcal{F}) \longrightarrow H^1_A(X, \mathcal{F}) \longrightarrow H^1_A(X, \mathcal{L}_0^\vee).$$

Since $\operatorname{depth}_A(\mathcal{O}_X) \geq 2$, the outer groups vanish, and we conclude $\operatorname{depth}_A(\mathcal{F}) \geq 2$. Consequently, $\mathcal{F}$ is almost locally free. By the same argument, we see that $\mathcal{G}^{\vee\vee}$ is almost locally free. Since the canonical map $\mathcal{G} \to \mathcal{G}^{\vee\vee}$ is bijective on some thick open subset, we conclude that it is bijective on Y, hence $\mathcal{G} \simeq i^*(\mathcal{F})$.

It remains to check that $\mathcal{G} \mapsto i_*(\mathcal{G})$ is the desired inverse equivalence. Extend $\mathcal{G}$ to an almost locally free $\mathcal{O}_X$-module $\mathcal{F}$. Since the adjunction map $\mathcal{F} \to i_* i^*(\mathcal{F})$ is bijective, we are done. □

Remark 1.2. Given two almost locally free sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$, the sheaf $\mathcal{F} = \mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)$ is almost locally free as well. This is because, by Lemma 1.1, the middle map in (1) is bijective, so that the cohomology groups with support vanish. As a consequence, almost locally free sheaves are reflexive.

An Azumaya algebra $\mathcal{A}$ is called *almost trivial* if its Brauer class $\operatorname{cl}(\mathcal{A}) \in \operatorname{Br}(X)$ vanishes on some thick open subset. This easily implies that $\mathcal{A} \simeq \mathcal{E}nd(\mathcal{F})$ for some almost locally free sheaf $\mathcal{F}$. However, the condition that the $\mathcal{O}_X$-algebra $\mathcal{E}nd(\mathcal{F})$ is an Azumaya algebra implies more.

Definition 1.3. A coherent $\mathcal{O}_X$-module $\mathcal{F}$ is called *balanced* if for each geometric point $\bar{x} \to X$, there is a decomposition $\mathcal{F}_{\bar{x}} \simeq \bigoplus_{i=1}^r \mathcal{L}_{\bar{x}}$ with $r > 0$ for some almost invertible $\mathcal{O}_{X,\bar{x}}$-module $\mathcal{L}_{\bar{x}}$.

Here $\mathcal{O}_{X,\bar{x}}$ is the strict henselization of the local ring $\mathcal{O}_{X,x}$. Perhaps it goes without saying that *almost invertible sheaves* are invertible on a thick open subset and have depth ≥ 2 outside. By fpqc-descent, balanced sheaves are almost locally free. They are closely related to Azumaya algebras, and the following result reduces the existence of certain Azumaya algebras to the existence of balanced sheaves.

Proposition 1.4. *Let $\mathcal{F}$ be a balanced $\mathcal{O}_X$-module. Then $\mathcal{E}nd(\mathcal{F})$ is an almost trivial Azumaya algebra, and each almost trivial Azumaya algebra has this form.*

Proof. Obviously, $\mathcal{A} = \mathcal{E}nd(\mathcal{F})$ is a trivial Azumaya algebra on some thick open subset. To check that it is an Azumaya algebra on X, we may assume that X is strictly local, and that $\mathcal{F} = \bigoplus_{i=1}^r \mathcal{L}$ for some almost invertible sheaf $\mathcal{L}$. Being bijective on some thick open subset, the map $\mathcal{O}_X \to \mathcal{E}nd(\mathcal{L})$ is everywhere bijective by Lemma 1.1. Consequently, $\mathcal{A} \simeq \mathcal{M}at_r(\mathcal{O}_X)$ is an Azumaya algebra.

Conversely, let $\mathcal{A}$ be an almost trivial Azumaya algebra. Choose a thick open subset $i : U \subset X$ on which the Brauer class is trivial. Then there is an isomorphism $\mathcal{A}_U \to \mathcal{E}nd(\mathcal{G})$ for some locally free $\mathcal{O}_U$-module $\mathcal{G}$. By Lemma 1.1, this induces an isomorphism of algebras $\mathcal{A} \to \mathcal{E}nd(\mathcal{F})$, where $\mathcal{F} = i_*(\mathcal{G})$.

If remains to check that the almost locally free sheaf $\mathcal{F}$ is balanced. To do so, we may assume that X is strictly local, so $\mathcal{A} = \mathcal{M}at_r(\mathcal{O}_X)$. Now $\mathcal{F}$ is a module over $\mathcal{M}at_r(\mathcal{O}_X)$. By Morita equivalence (see e.g. [19] p. 53), $\mathcal{F} = \bigoplus_{i=1}^r \mathcal{M}$ for some

coherent $\mathcal{O}_X$-module $\mathcal{M}$. Clearly, $\mathcal{M}$ is invertible on a thick open subset and has depth ≥ 2 outside. In other words, $\mathcal{M}$ is almost invertible. $\square$

Next, we shall generalize some notions from Hartshorne's paper on generalized divisors [15]. For simplicity, we assume that X satisfies Serre's condition (S_2), such that the points of codimension one are precisely the points of depth one. Set

$$\mathcal{ADiv}_X = \bigoplus_{x\in X^{(1)}} (i_x)_*(\operatorname{Div}(\mathcal{O}_{X,x}))$$

where the sum runs over all points of codimension one. The elements of the group $\operatorname{ADiv}(X) = \Gamma(X, \mathcal{ADiv}_X)$ are called *almost Cartier divisors*. For normal schemes, $\mathcal{ADiv}_X$ is just the sheaf of Weil divisors. As in the normal case, an almost Cartier divisor $D \in \operatorname{ADiv}(X)$ defines an almost invertible sheaf $\mathcal{O}_X(D)$, which is invertible in codimension one and satisfies Serre's condition (S_2). According to [11] Proposition 21.1.8, the canonical map $\mathcal{Div}_X \to \mathcal{ADiv}_X$ is injective. The exact sequence

$$0 \longrightarrow \mathcal{Div}_X \longrightarrow \mathcal{ADiv}_X \longrightarrow \mathcal{P}_X \longrightarrow 0$$

defines an abelian sheaf $\mathcal{P}_X$ on the étale site $X_{\text{ét}}$. For a geometric point $\bar{x} \to X$ with corresponding strict localization $\mathcal{O}_{X,\bar{x}} = \mathcal{O}^{\text{sh}}_{X,x}$, the stalk is

$$\mathcal{P}_{X,\bar{x}} = \operatorname{ADiv}(\mathcal{O}_{X,\bar{x}})/\operatorname{Div}(\mathcal{O}_{X,\bar{x}}).$$

For normal schemes, this reduces to the class group $\operatorname{Cl}(\mathcal{O}_{X,\bar{x}})$. The preceding short exact sequence gives a long exact sequence in étale cohomology

$$\operatorname{ADiv}(X) \to H^0(X,\mathcal{P}) \longrightarrow H^1(X,\mathcal{Div}_X) \longrightarrow H^1(X,\mathcal{ADiv}_X).$$

We also have an exact sequence

$$0 \longrightarrow H^1(X,\mathcal{Div}_X) \longrightarrow H^2(X,\mathbb{G}_m) \longrightarrow H^2(X^{(0)},\mathbb{G}_m),$$

where $X^{(0)} = \coprod \operatorname{Spec}(\mathcal{O}_{X,\eta})$ is the scheme of generic points, and you easily infer that the classes of almost trivial Azumaya algebra lie in the image of the iterated coboundary map

$$H^0(X,\mathcal{P}) \longrightarrow H^1(X,\mathcal{Div}_X) \longrightarrow H^2(X,\mathbb{G}_m). \tag{3}$$

If $\mathcal{F}$ is an almost locally free sheaf with decompositions $\mathcal{F}_{\bar{x}} = \bigoplus_{i=1}^r \mathcal{L}_{\bar{x}}$, the function

$$\bar{x} \mapsto \operatorname{cl}(\mathcal{L}_{\bar{x}}) \in \operatorname{ADiv}(\mathcal{O}_{X,\bar{x}})/\operatorname{Div}(\mathcal{O}_{X,\bar{x}}) = \mathcal{P}_{X,\bar{x}}$$

depends only on $\mathcal{F}$ and yields a section $s_{\mathcal{F}} \in \Gamma(X,\mathcal{P}_X)$. The following result is due to Gabber:

Proposition 1.5 (Gabber)**.** *Let $\mathcal{F}$ be an balanced $\mathcal{O}_X$-module. Then the class of the Azumaya algebra $\mathcal{E}nd(\mathcal{F})$ in $H^2(X,\mathbb{G}_m)$ is the inverse of the image of the section $s_{\mathcal{F}} \in H^0(X,\mathcal{P}_X)$ under the iterated coboundary map in (3).*

Proof. Set $\mathcal{A} = \mathcal{E}nd(\mathcal{F})$. According to [7] p. 341, its cohomology class in $H^2(X,\mathbb{G}_m)$ is given by the gerbe $d(\mathcal{A})$ of trivializations for $\mathcal{A}$, which associates to each étale $U \to X$ the groupoid of pairs $(\mathcal{E},\varphi)$, where $\mathcal{E}$ is a locally free $\mathcal{O}_U$-module, and $\varphi : \mathcal{E}nd(\mathcal{E}) \to \mathcal{A}_U$ is an isomorphism. The action $\mathbb{G}_m \to \mathcal{A}ut(\mathcal{E},\varphi)$ is given by homotheties on $\mathcal{E}$.

Set $\mathcal{M}_X^\times = g_*g^*(\mathbb{G}_m)$, where $g : X^{(0)} \to X$ is the inclusion of generic points, and let $\pi : \mathcal{M}_X^\times \to \mathcal{Div}_X$ and $p : \mathcal{ADiv}_X \to \mathcal{P}_X$ be the natural surjections. Then the image of $s_{\mathcal{F}} \in H^0(X,\mathcal{P}_X)$ under the iterated coboundary map is given by the gerbe of $\mathcal{M}_X^\times$-liftings of the $\mathcal{Div}_X$-torsor $p^{-1}(s_{\mathcal{F}})$. This gerbe associates to each étale

$U \to X$ the groupoid of pairs $(\mathcal{T}, \psi)$, where $\mathcal{T}$ is a $\mathcal{M}_U^\times$-torsor, and $\psi : \mathcal{T} \to p^{-1}(s_\mathcal{F})$ is a π-morphism of torsors. Moreover, the action $\mathbb{G}_m \to \mathcal{A}ut(\mathcal{T}, \psi)$ is given by translation on $\mathcal{T}$.

We have to construct an equivalence between the preceding two stacks that is equivariant for the sign change map $-1 : \mathbb{G}_m \to \mathbb{G}_m$. First note that the stack of pairs $(\mathcal{E}, \varphi)$ is $\mathbb{G}_m$-equivalent to the stack of triples $(\mathcal{E}, \mathcal{L}, \psi)$, where $\mathcal{E}$ is a locally free $\mathcal{O}_U$-module, and $\mathcal{L}$ is an almost invertible $\mathcal{O}_U$-module, and $\psi : \mathcal{E} \otimes \mathcal{L} \to \mathcal{F}$ is an isomorphism. The action $\mathbb{G}_m \to \mathcal{A}ut(\mathcal{E}, \mathcal{L}, \psi)$ is given by homotheties on $\mathcal{E}$ and inverse homotheties on $\mathcal{L}$.

Obviously, the local section $s_\mathcal{L} \in \Gamma(U, \mathcal{P}_X)$ is nothing but the restriction of the global section $s_\mathcal{F} \in \Gamma(X, \mathcal{P}_X)$. Now let $\mathcal{M}_U^\times(\mathcal{L})$ be the sheaf of invertible meromorphic sections in $\mathcal{L}|_U$, which is a $\mathcal{M}_U^\times$-torsor. Dividing out the induced $\mathbb{G}_m$-action, we obtain a π-morphism

$$\varphi : \mathcal{M}_U^\times(\mathcal{L}) \longrightarrow p^{-1}(s_\mathcal{L}) = p^{-1}(s_\mathcal{F}|_U).$$

Consequently, the functor $(\mathcal{E}, \mathcal{L}, \psi) \mapsto (\mathcal{M}_U^\times(\mathcal{L}), \varphi)$ gives the desired antiequivariant equivalence of $\mathbb{G}_m$-gerbes. □

For the rest of the section, we assume that X satisfies Serre's condition (S_1), that is, X has no embedded components. Let $g : X^{(0)} \to X$ be the inclusion of the generic points. We can relate generically trivial Brauer classes to torsors of Cartier divisors as follows. The exact sequence

$$0 \longrightarrow \mathbb{G}_m \longrightarrow g_* g^*(\mathbb{G}_m) \longrightarrow \mathcal{D}iv_X \longrightarrow 0,$$

together with $R^1 g_*(\mathbb{G}_{m,X^{(0)}}) = 0$ and $\mathrm{Pic}(X^{(0)}) = 0$, gives an exact sequence

$$0 \longrightarrow H^1(X, \mathcal{D}iv_X) \longrightarrow H^2(X, \mathbb{G}_m) \longrightarrow H^2(X^{(0)}, \mathbb{G}_m). \tag{4}$$

Hence each $\alpha \in \mathrm{Br}(X)$ with $g^*(\alpha) = 0$ comes from a $\mathcal{D}iv_X$-torsor. To make this explicit, choose an Azumaya algebra $\mathcal{A}$, say of rank r, representing the class α, and let $f : B \to X$ be the associated *Brauer–Severi* scheme. This is the $\mathbb{P}^{r-1}$-bundle

$$B = \mathrm{Isom}(\mathcal{M}at_r(\mathcal{O}_X), \mathcal{A}) \times_{\mathrm{PGL}_r} \mathbb{P}^{r-1}$$

on the étale site $X_{\text{ét}}$. Here we use the left PGL_r-action coming from the canonical representation $\mathrm{PGL}_r \to \mathrm{Aut}(\mathbb{P}^{r-1})$ described in [21] Chap. 0 §5. Set $P = B \times_X X^{(0)}$, and pick an invertible $\mathcal{O}_P$-module $\mathcal{O}_P(1)$ of fiber degree one. This leads to a $\mathcal{D}iv_X$-torsor $\mathcal{T}$ as follows. Define

$$\Gamma(X, \mathcal{T}) = \{\mathrm{cl}(\mathcal{L}, t)\},$$

where $\mathcal{L}$ is an invertible $\mathcal{O}_B$-module, and $t : \mathcal{O}_P(1) \to \mathcal{L}|_P$ is an isomorphism. Here $\mathrm{cl}(\mathcal{L}, t)$ denotes isomorphism class, and two pairs $(\mathcal{L}, t)$ and $(\mathcal{L}', t')$ are called isomorphic if there is an isomorphism $\phi : \mathcal{L} \to \mathcal{L}'$ with $t \circ \phi = t'$. Define $\mathcal{T}$ in the same way on the étale site.

By [11] Proposition 21.2.11, the sections of $\mathcal{D}iv_X$ correspond to $\mathrm{cl}(\mathcal{M}, s)$, where $\mathcal{M}$ is an invertible $\mathcal{O}_X$-module, and $s : \mathcal{O}_{X^{(0)}} \to \mathcal{M}|_{X^{(0)}}$ is a trivialization. The map

$$(\mathcal{M}, s), (\mathcal{L}, t) \mapsto (f^*(\mathcal{M}) \otimes \mathcal{L}, f^*(s) \otimes t)$$

turns $\mathcal{T}$ into a $\mathcal{D}iv_X$-torsor.

Proposition 1.6. *The Brauer class* $\mathrm{cl}(\mathcal{A}) \in H^2(X, \mathbb{G}_m)$ *is the opposite for the image of the torsor class* $\mathrm{cl}(\mathcal{T}) \in H^1(X, \mathcal{D}iv_X)$ *under the coboundary map in (4).*

Proof. As in the proof of Proposition 1.5, the Brauer class is given by the gerbe of trivializations $\varphi : \mathcal{E}nd(\mathcal{E}) \to \mathcal{A}$. This gerbe is equivalent to the gerbe of trivializations $(\mathcal{E}, u)$, where $u : B \to \mathbb{P}(\mathcal{E}^\vee)$ is an isomorphism. According to [7] Chap. V Lemma 4.8.1, the latter gerbe is antiequivalent to the $\mathbb{G}_m$-gerbe of invertible $\mathcal{O}_P$-modules $\mathcal{L}$ of fiber degree one.

Set $\mathcal{M}_X^\times = g_* g^* \mathbb{G}_m$. The image of the torsor class $\mathrm{cl}(\mathcal{T}) \in H^1(X, \mathcal{D}iv_X)$ is the gerbe of $\mathcal{M}_X^\times$-liftings $\psi : \mathcal{S} \to \mathcal{T}$ for the $\mathcal{D}iv_X$-torsor $\mathcal{T}$. Given an invertible $\mathcal{O}_P$-module $\mathcal{L}$ of fiber degree one, we obtain an $\mathcal{M}_X^\times$-torsor

$$\mathcal{S} = \left\{ (\mathcal{L}, t) \mid t : \mathcal{O}_P(1) \xrightarrow{\simeq} \mathcal{L}_P \right\},$$

where the action is by multiplication on t. Clearly, the quotient $\mathcal{S}/\mathbb{G}_m = \{\mathrm{cl}(\mathcal{L}, t)\}$ is canonically isomorphic to $\mathcal{T}$, so we obtain a morphism of torsors $\psi : \mathcal{S} \to \mathcal{T}$. To see that $\mathcal{L} \mapsto (\mathcal{S}, \psi)$ is a $\mathbb{G}_m$-equivalence, note that a $\mathcal{M}_X^\times$-lifting of the torsor $\mathcal{T}$ exists if and only if $\mathcal{T}$ is trivial, because $H^1(X, \mathcal{M}_X^\times) = 0$. □

2. Generically trivial Brauer classes

In this section, we turn to normal surfaces. The task is to prove the following result, which is a major step towards showing $\mathrm{Br}(X) = \mathrm{Br}'(X)$.

Proposition 2.1. *Let X be a separated normal algebraic surface. Then* $\mathrm{Br}(X)$ *contains each class $\alpha \in \mathrm{Br}'(X)$ that is generically trivial.*

Proof. We start with some preliminary reductions. By [6] Chap. II Lemma 4, we may assume that the ground field k is separably closed. Since X is separated and of finite type, the Nagata Compactification Theorem [24] gives a compactification $X \subset \bar{X}$. By resolution of singularities, we may assume that $\mathrm{Sing}(X) = \mathrm{Sing}(\bar{X})$. As in [12] Chap. II Theorem 2.1, each generically trivial Brauer class $\alpha \in \mathrm{Br}'(X)$ extends to $\mathrm{Br}'(\bar{X})$, so we may begin the proof with the additional assumption that X is proper.

As discussed in Section 1, the exact sequence

$$0 \longrightarrow H^1(X, \mathcal{D}iv_X) \to H^2(X, \mathbb{G}_m) \longrightarrow H^2(X^{(0)}, \mathbb{G}_m)$$

shows that our cohomology class $\alpha \in \mathrm{Br}'(X)$ lies in $H^1(X, \mathcal{D}iv_X)$. The exact sequence

$$0 \longrightarrow \mathcal{D}iv_X \longrightarrow \mathcal{Z}_X^1 \longrightarrow \mathcal{P}_X \longrightarrow 0$$

yields an exact sequence

$$Z^1(X) \longrightarrow H^0(X, \mathcal{P}_X) \longrightarrow H^1(X, \mathcal{D}iv_X) \longrightarrow 0.$$

Choose a global section $s \in H^0(X, \mathcal{P}_X)$ mapping to α. Since $r\alpha = 0$ for some integer $r > 0$, there is a global Weil divisor $E \in Z^1(X)$ mapping to the section $rs \in H^0(X, \mathcal{P}_X)$.

The sheaf $\mathcal{P}_X$ is supported by the singular locus $\mathrm{Sing}(X)$. For each singular point $x \in X$, the stalk is $\mathcal{P}_{X,x} = \mathrm{Cl}(\mathcal{O}_{X,x}^h)$, where $\mathcal{O}_{X,x} \subset \mathcal{O}_{X,x}^h$ is the henselization (which is the strict localization, because k is separably closed). Suppose $x_1, \ldots, x_m \in X$ are the singularities, and set

$$X^h = \mathrm{Spec}(\prod_{i=1}^m \mathcal{O}_{X,x_i}^h) = \coprod_{i=1}^m \mathrm{Spec}(\mathcal{O}_{X,x_i}^h).$$

Then $H^0(X,\mathcal{P}_X)=\mathrm{Cl}(X^h)$. Choose a Weil divisor $D\in Z^1(X^h)$ representing the section $s\in H^0(X,\mathcal{P}_X)$, such that $E|_{X^h}\sim rD$. According to Proposition 1.5, it suffices to construct a reflexive $\mathcal{O}_X$-module $\mathcal{F}$ with

$$\mathcal{F}\otimes\mathcal{O}_{X^h}=\bigoplus_{i=1}^r\mathcal{O}_{X^h}(-D),$$

for then $\mathcal{A}=\mathcal{E}nd(\mathcal{F})$ would be the desired Azumaya algebra.

Let $f:Y\to X$ be a resolution of singularities, and $Y_0\subset Y$ be the reduced exceptional curve. Set $Y^h=Y\times_X X^h$. The following crucial result, which is due to Gabber, tells us that certain vector bundles on Y^h are already determined on suitable infinitesimal neighborhoods of Y_0.

Lemma 2.2 (Gabber)**.** *Let $\mathcal{B}$ be a family of locally free $\mathcal{O}_{Y_0}$-modules of fixed rank $n\geq 0$. Suppose that $\mathcal{B}$ is, up to tensoring with line bundles, a bounded family. Then there is an exceptional curve $R\subset Y$ so that the map $H^1(Y^h,\mathrm{GL}_n)\to H^1(R,\mathrm{GL}_n)$ is bijective on the subsets of vector bundles whose restriction to Y_0 lies in $\mathcal{B}$.*

Proof. Let $\mathcal{I}\subset\mathcal{O}_Y$ be the ideal of $Y_0\subset Y$. Since the intersection matrix for the irreducible components of Y_0 is negative definite, there is an exceptional curve $A\subset Y$ with support Y_0 so that $\mathcal{O}_{Y_0}(-A)$ is ample. Then $\mathcal{O}_A(-A)$ is ample as well. Since the family $\{\mathcal{E}nd(\mathcal{E})\mid\mathcal{E}\in\mathcal{B}\}$ is bounded, there is an integer $m_0>0$ so that

$$H^1(A,\mathcal{E}nd(\mathcal{E})\otimes\mathcal{O}_A(-mA))=0 \tag{5}$$

for all $m\geq m_0$ and all $\mathcal{E}\in\mathcal{B}$.

Let $\mathfrak{Y}\subset Y$ be the formal completion along $Y_0\subset Y$. We first check the statement of the Lemma for the formal scheme $\mathfrak{Y}$ instead of Y^h. Note that the canonical map

$$H^1(\mathfrak{Y},\mathrm{GL}_n)\longrightarrow\varprojlim H^1(mA,\mathrm{GL}_n)$$

is bijective, as explained in [1], proof of Theorem 3.5. Let $\mathcal{J}\subset\mathcal{O}_Y$ be the ideal of $A\subset Y$. The obstruction to lifting a vector bundle $\mathcal{E}$ on mA to $(m+1)A$ lies in

$$H^2(R,\mathcal{E}nd(\mathcal{E})\otimes\mathcal{J}^m/\mathcal{J}^{m+1})=0,$$

so the restriction maps $H^1(\mathfrak{Y},\mathrm{GL}_n)\to H^1(mA,\mathrm{GL}_n)$ are surjective for all $m\geq 0$.

Fix an integer $m\geq m_0$, and let $\mathcal{E},\mathcal{E}'$ be two vector bundles on $(m+1)A$ that are isomorphic on mA and whose restrictions to Y_0 belong to the family $\mathcal{B}$. Choose an isomorphism $\psi:\mathcal{E}|_{mA}\to\mathcal{E}'|_{mA}$. Locally on $(m+1)A$, we can lift this isomorphism to an isomorphism $\mathcal{E}\to\mathcal{E}'$. The sheaf of such liftings is a torsor under

$$\mathcal{H}om(\mathcal{E},\mathcal{E}')\otimes\mathcal{J}^m/\mathcal{J}^{m+1}\simeq\mathcal{E}nd(\mathcal{E})\otimes\mathcal{J}^m/\mathcal{J}^{m+1}\simeq\mathcal{E}nd(\mathcal{E})\otimes\mathcal{O}_A(-mA).$$

This sheaf has no first cohomology by (5). The upshot is that a global lifting of $\psi:\mathcal{E}|_{mA}\to\mathcal{E}'|_{mA}$ exists. Consequently, for $R=m_0A$, the mapping $H^1(\mathfrak{Y},\mathrm{GL}_n)\to H^1(R,\mathrm{GL}_n)$ is bijective on the subsets of vector bundles whose restriction belongs to the family $\mathcal{B}$.

Finally, we pass to Y^h. By the Artin Approximation Theorem ([1] Thm. 3.5), the map $H^1(Y^h,\mathrm{GL}_n)\to H^1(\mathfrak{Y},\mathrm{GL}_n)$ is injective and has dense image. So given a formal vector bundle $\mathcal{E}$ with $\mathcal{E}|_{Y_0}\in\mathcal{B}$, we find a vector bundle $\mathcal{E}^h$ on Y^h with $\mathcal{E}^h|_R\simeq\mathcal{E}|_R$. By the choice of R, this implies $\mathcal{E}^h|_{\mathfrak{Y}}\simeq\mathcal{E}$. ∎

We proceed with the proof of Proposition 2.1. Let $\mathcal{B}$ be the family of vector bundles on Y_0 of rank r that are free up to tensoring with line bundles, and choose an exceptional divisor $R\subset Y$ as in the preceding Lemma. Let $D'\in Z^1(Y^h)$ be

the strict transform of $D \in Z^1(X^h)$. Let $E' \in Z^1(Y)$ be the unique Weil divisor that is the strict transform of $E \in Z^1(X)$ on $Y - Y_0$, and satisfies $E' \sim rD'$ on Y^h. Set $\mathcal{L} = \mathcal{O}_R(D')$. According to Lemma 2.2, we have to construct a locally free $\mathcal{O}_Y$-module $\mathcal{E}$ with $\mathcal{E}_R \simeq \bigoplus_{i=1}^r \mathcal{L}^\vee$. For then the double dual $\mathcal{F} = f_*(\mathcal{E})^{\vee\vee}$ would be the desired reflexive $\mathcal{O}_X$-module. We shall construct such a vector bundle as an elementary transformation of the trivial bundle $\mathcal{O}_Y^{\oplus r}$.

Choose an ample divisor $A \subset Y$. Replacing the divisors D and E by $D + f_*(tA)$ and $E + f_*(rtA)$, respectively, does not change the class $\alpha \in \mathrm{Br}'(X)$. Choosing $t \gg 0$, we may assume that $\mathcal{L} = \mathcal{O}_R(D')$ is very ample, and that $H^1(Y, \mathcal{O}_Y(E' - R)) = 0$. Next, choose pairwise disjoint effective Cartier divisors $D_1, \ldots, D_r \subset R$, each one representing $\mathcal{L}$. Let $s_i \in \Gamma(R, \mathcal{L})$ be the corresponding sections. Regard their product $s_1 \otimes \ldots \otimes s_r$ as a section of $\mathcal{O}_R(E')$. By construction, the group on the right in the exact sequence

$$H^0(Y, \mathcal{O}_Y(E')) \longrightarrow H^0(R, \mathcal{O}_R(E')) \longrightarrow H^1(Y, \mathcal{O}_Y(E' - R))$$

is zero. Consequently, $E' \in Z^1(Y)$ is linearly equivalent to an effective divisor $H \subset Y$ with $H \cap R = D_1 \cup \ldots \cup D_r$. Now choose a closed subset $S \subset H - R$ so that each Cartier divisor $D_i \subset H$ is principal on $H - S$. For each $1 \leq i \leq r$, choose an exact sequence

$$0 \longrightarrow \mathcal{O}_H(C_i) \xrightarrow{t_i} \mathcal{O}_H \longrightarrow \bigoplus_{j \neq i} \mathcal{O}_{D_j} \longrightarrow 0 \tag{6}$$

for certain Cartier divisors $C_i \in \mathrm{Div}(H)$ supported by S. As explained in [23], p. 152, it suffices to construct the desired Azumaya $\mathcal{O}_X$-algebra on $X - f(S)$. Hence it suffices to construct the desired locally free $\mathcal{O}_Y$-module $\mathcal{E}$ on $Y - S$, and we may replace X, Y by the complements $X - f(S)$, $Y - S$, respectively. Now the exact sequence (6) induces an exact sequence

$$\mathcal{O}_Y \xrightarrow{t_i} \mathcal{O}_H \longrightarrow \bigoplus_{j \neq i} \mathcal{O}_{D_j} \longrightarrow 0.$$

The map $t = (t_1, \ldots, t_r) : \bigoplus_{i=1}^r \mathcal{O}_Y \longrightarrow \mathcal{O}_H$ is surjective, because the D_i are pairwise disjoint. The exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_{i=1}^r \mathcal{O}_Y \xrightarrow{t} \mathcal{O}_H \longrightarrow 0$$

defines a locally free $\mathcal{O}_Y$-module $\mathcal{E}$, because the cokernel $\mathcal{O}_H$ has homological dimension $\mathrm{hd}(\mathcal{O}_H) = 1$. Restricting to the curve $R \subset Y$, we obtain an exact sequence

$$\mathcal{T}or^1_{\mathcal{O}_Y}(\mathcal{O}_H, \mathcal{O}_R) \longrightarrow \mathcal{E}_R \longrightarrow \bigoplus_{i=1}^r \mathcal{O}_R \xrightarrow{t_R} \mathcal{O}_{H \cap R} \longrightarrow 0.$$

The term on the left is zero, because the curves $H, R \subset Y$ have no common components. By construction, $t_i|_{D_j} = 0$ for $i \neq j$, so the induced surjection t_R is a diagonal matrix of the form

$$t_R = \begin{pmatrix} t_1|_{D_1} & & 0 \\ & \ddots & \\ 0 & & t_r|_{D_r} \end{pmatrix} : \bigoplus_{i=1}^r \mathcal{O}_R \longrightarrow \bigoplus_{i=1}^r \mathcal{O}_{D_i} = \mathcal{O}_{H \cap R}.$$

Consequently $\mathcal{E}_R \simeq \bigoplus_{i=1}^r \mathcal{L}^\vee$. Hence $\mathcal{E}$ is the desired locally free $\mathcal{O}_Y$-module. $\square$

Remark 2.3. In my first proof of Proposition 2.1, I used a result of Treger ([27] Prop. 3.5), which states that the map $H^1(Y^h, \mathrm{GL}_n) \to H^1(R, \mathrm{GL}_n)$ is bijective for a suitable exceptional curve $R \subset Y$. As Gabber pointed out, this statement is wrong for $n \geq 2$. His counterexample goes as follows.

Let $X = \mathrm{Spec}(A)$ be a complete normal local surface singularity, and $f : Y \to X$ a resolution of singularity, and $Y_m \subset Y$ the infinitesimal neighborhoods of the exceptional curve Y_0. Suppose $n = 2$ for simplicity, and assume there is an exceptional curve R as above. Fix an ample invertible $\mathcal{O}_Y$-module $\mathcal{L}$, set $\mathcal{L}_m = \mathcal{L}|_{Y_m}$, and choose $m \geq 0$ with $R \subset Y_m$. The exact sequence

$$0 \longrightarrow \mathcal{L}_0^{-t}(-Y_m) \longrightarrow \mathcal{L}_{m+1}^{-t} \longrightarrow \mathcal{L}_m^{-t} \longrightarrow 0$$

yields an exact sequence

$$H^0(Y_m, \mathcal{L}_m^{-t}) \to H^1(Y_0, \mathcal{L}_0^{-t}(-Y_m)) \to H^1(Y_{m+1}, \mathcal{L}_{m+1}^{-t}) \to H^1(Y_m, \mathcal{L}_m^{-t}) \to 0.$$

For $t \gg 0$, the group on the left is zero, and $H^1(Y_0, \mathcal{L}_0^{-t}(-Y_m))$ is nonzero. It follows that there is a nonzero class $\zeta \in H^1(Y, \mathcal{L}^{-t})$ restricting to zero in $H^1(Y_m, \mathcal{L}_m^{-t})$. This defines a nonsplit extension

$$0 \longrightarrow \mathcal{L}^{-t} \longrightarrow \mathcal{E} \xrightarrow{\psi} \mathcal{O}_Y \longrightarrow 0, \tag{7}$$

which splits on R. By the defining property of the curve R, there is a bijection $\phi : \mathcal{O}_Y \oplus \mathcal{L}^{-t} \to \mathcal{E}$. The composition

$$\mathcal{O}_Y \xrightarrow{\phi} \mathcal{E} \xrightarrow{\psi} \mathcal{O}_Y$$

is surjective on Y_m, because $H^0(Y_m, \mathcal{L}_m^{-t}) = 0$. By the Nakayama Lemma, the composition is surjective on the formal completion, and hence on Y as well. So the extension (7) splits, contradicting $\zeta \neq 0$.

3. Elementary transformations of Brauer–Severi schemes

We come to the main result of this paper.

Theorem 3.1. *Let X be a separated geometrically normal algebraic surface. Then we have* $\mathrm{Br}(X) = \mathrm{Br}'(X)$.

Proof. According to [6] Chap. II Lemma 4, we may assume that the ground field is algebraically closed. Fix a class $\alpha \in \mathrm{Br}'(X)$. In light of Proposition 2.1, it suffices to construct an Azumaya algebra representing $\alpha \in \mathrm{Br}'(X)$ generically. Choose a resolution of singularities $f : Y \to X$, and let $\mathfrak{Y} \subset Y$ be the formal completion along the reduced exceptional curve $Y_0 \subset Y$. According to [12] Theorem 2.1, there is an Azumaya $\mathcal{O}_Y$-algebra $\mathcal{A}$ representing $f^*(\alpha)$. The task now is to choose such an Azumaya algebra so that the formal vector bundle $\mathcal{A}|_{\mathfrak{Y}}$ is trivial. For then, as explained in [23] p. 152, the $\mathcal{O}_X$-algebra $f_*(\mathcal{A})$ is an Azumaya algebra, which represents $\alpha \in \mathrm{Br}'(X)$ generically. Note that we may remove finitely many closed smooth points from X.

First, we check that the formal Azumaya algebra $\mathcal{A}|_{\mathfrak{Y}}$ is trivial. Since the ground field is separably closed, there is a locally free $\mathcal{O}_{Y_0}$-module $\mathcal{E}_0$ and an isomorphism $\varphi_0 : \mathcal{E}nd(\mathcal{E}_0) \to \mathcal{A}|_{Y_0}$. The following argument due to Gabber shows that the pair $(\mathcal{E}_0, \varphi_0)$ extends over all infinitesimal neighborhoods $Y_0 \subset Y_n$. Let $d(\mathcal{A}|_{Y_n})$ be the $\mathbb{G}_m$-gerbe of trivializations of $\mathcal{A}|_{Y_n}$. The restriction map gives a cartesian functor $d(\mathcal{A}|_{Y_{n+1}}) \to d(\mathcal{A}|_{Y_n})$. Consider the corresponding stack of liftings of trivializations. This is a gerbe for the abelian sheaf $\mathcal{I}^{n+1}/\mathcal{I}^{n+2}$ on the étale site of Y_0,

where $\mathcal{I} = \mathcal{O}_Y(-Y_0)$. Since $H^2(Y_0, \mathcal{I}^{n+1}/\mathcal{I}^{n+2}) = 0$, the gerbe of liftings is trivial. Consequently, we have $\mathcal{A}|_{\mathfrak{Y}} \simeq \mathcal{E}nd(\mathcal{E})$ for some locally free $\mathcal{O}_{\mathfrak{Y}}$-module $\mathcal{E}$.

The idea now is to make an elementary transformation along a curve $H \subset Y$, so that $\mathcal{E}$ becomes free on certain infinitesimal neighborhood $Y_0 \subset R$. Furthermore, we shall choose the curve R so that the freeness of $\mathcal{E}_R$ implies the freeness of $\mathcal{E}$. This requires some preparation. Let me introduce three numbers m, k, q depending on Y and $\mathcal{E}$. First, set $m = \operatorname{rank}(\mathcal{E})$. Second, let $k \geq 1$ be the order of the cokernel for the map $\operatorname{Pic}(Y) \to \operatorname{NS}(Y_0)$ onto the Néron–Severi group. Third, define $q = 1$ in characteristic zero. In characteristic $p > 0$, let $q > 0$ be a p-th power so that the unipotent part of $\operatorname{Pic}^0(\mathfrak{Y})$ is q-torsion. This works, because $\operatorname{Pic}^0_{\mathfrak{Y}}$ is an algebraic group scheme.

Now set $r = mkq$, and let $\mathcal{B}$ be the family of locally free $\mathcal{O}_{Y_0}$-modules of rank r which are free up to tensoring with line bundles. Choose a curve $R \subset Y$ as in Lemma 2.2. Finally, let $X \subset \bar{X}$ be a compactification with $\operatorname{Sing}(X) = \operatorname{Sing}(\bar{X})$, and let $Y \subset \bar{Y}$ be the corresponding compactification.

Claim. *We can modify the Azumaya algebra $\mathcal{A}$ and the vector bundle $\mathcal{E}$ so that $\mathcal{E}$ is a globally generated formal vector bundle of rank r, and that there is a very ample invertible $\mathcal{O}_{\bar{Y}}$-module $\mathcal{L}$ with $\det(\mathcal{E}) = \mathcal{L}_{\mathfrak{Y}}$ and $H^1(\bar{Y}, \mathcal{L}(-R)) = 0$.*

Proof. Tensoring $\mathcal{E}$ with an ample line bundle, we archive that $\mathcal{E}$ is globally generated and that $\det(\mathcal{E})$ is ample. Next, we replace the Azumaya algebra $\mathcal{A}$ by $\mathcal{A} \otimes \mathcal{E}nd(\mathcal{O}_Y^{\oplus k})$. This replaces the vector bundle $\mathcal{E}$ by $\mathcal{E}^{\oplus k}$, and $\det(\mathcal{E})$ by $\det(\mathcal{E})^{\otimes k}$. Consequently, we may assume that $\det(\mathcal{E})$ is numerically equivalent to some ample invertible $\mathcal{O}_{\mathfrak{Y}}$-module $\mathcal{L}_{\mathfrak{Y}}$. Twisting $\mathcal{E}$ by a suitable power of $\mathcal{L}_{\mathfrak{Y}}$, we may assume that $\mathcal{L}_{\mathfrak{Y}}$ extends to an ample invertible $\mathcal{O}_{\bar{Y}}$-module $\mathcal{L}$ with $H^1(\bar{Y}, \mathcal{L}^{\otimes s}(-R)) = 0$ for all integers $s > 0$.

In characteristic zero, $\operatorname{Pic}^0(\mathfrak{Y})$ is a divisible group. Hence we find an invertible $\mathcal{O}_{\mathfrak{Y}}$-module $\mathcal{M}$ with $\mathcal{M}^{\otimes mk} \otimes \det(\mathcal{E}) \simeq \mathcal{L}_{\mathfrak{Y}}$. Replacing $\mathcal{E}$ by $\mathcal{E} \otimes \mathcal{M}$, we have $\det(\mathcal{E}) = \mathcal{L}_{\mathfrak{Y}}$. Now assume that we are in characteristic $p > 0$. Set $G = \operatorname{Pic}^0_{\mathfrak{Y}/k}$, and let $G' \subset G$ be the unipotent part. Then the quotient $G'' = G/G'$ is semiabelian, and $G''(k)$ is a divisible group. As in characteristic zero, we may assume that $\det(\mathcal{E}) \otimes \mathcal{L}_{\mathfrak{Y}}^{\vee}$ lies in the unipotent part of $\operatorname{Pic}^0(\mathfrak{Y})$, which is a q-torsion group. Passing to $\mathcal{A} \otimes \mathcal{E}nd(\mathcal{O}_X^{\oplus q})$ and $\mathcal{E}^{\oplus q}$, we are done. $\square$

We continue proving Theorem 3.1. Set $r = \operatorname{rank}(\mathcal{E})$ and $\Gamma = \Gamma(R, \mathcal{E}_R)$. The canonical surjection $\Gamma \otimes \mathcal{O}_R \to \mathcal{E}_R$ yields a morphism $\varphi : R \to \operatorname{Grass}_r(\Gamma)$ into the Grassmannian of r-dimensional quotients. Choose a generic r-dimensional subvector space $\Gamma' \subset \Gamma$. For each integer $k \geq 0$, let $G_k \subset \operatorname{Grass}_r(\Gamma)$ be the subscheme of surjections $\Gamma \to \Gamma''$ such that the composition $\Gamma' \to \Gamma''$ has rank $\leq k$. Note that G_{r-1} is a reduced Cartier divisor, and that G_{r-2} has codimension four (see [2], Sec. II.2).

By the dimensional part of Kleiman's Transversality Theorem ([18] Thm. 2), which is valid in all characteristics, the map $\varphi : R \to \operatorname{Grass}_r(\Gamma)$ is disjoint to G_{r-2} and passes through G_{r-1} in finitely many points. The upshot of this is that the quotient of the canonical map $\Gamma' \otimes \mathcal{O}_X \to \mathcal{E}$ is an invertible sheaf on some Cartier divisor $D \subset R$. Consequently, we have constructed an exact sequence

$$0 \longrightarrow \mathcal{O}_R^{\oplus r} \longrightarrow \mathcal{E}_R \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

In other words, the trivial vector bundle is the elementary transformation of $\mathcal{E}$ with respect to the surjection $\mathcal{E}_R \to \mathcal{O}_D$. In geometric terms: Blowing up $\mathbb{P}(\mathcal{E}_R)$ along

the section $\mathbb{P}(\mathcal{O}_D) \subset \mathbb{P}(\mathcal{E}_D)$ and contracting the strict transform of $\mathbb{P}(\mathcal{E}_D)$ yields $\mathbb{P}^{r-1}_R$ (see [22] Thm. 1.4).

We seek to extend this elementary transformation from the curve to the surface. Note that $\mathcal{L}_R = \det(\mathcal{E}_R) = \mathcal{O}_R(D)$. The exact sequence

$$H^0(\bar{Y}, \mathcal{L}) \longrightarrow H^0(R, \mathcal{L}_R) \longrightarrow H^1(\bar{Y}, \mathcal{L}(-R)) = 0$$

implies that $D = H \cap R$ for some ample curve $H \subset Y$. Since the ground field k is algebraically closed, Tsen's Theorem gives $H^2(H, \mathbb{G}_m) = 0$. Removing finitely many smooth points from the open subset $X \subset \bar{X}$, we may assume that $\mathcal{A}_H = \mathcal{E}nd(\mathcal{O}_H^{\oplus r})$. In other words, if $P \to Y$ is the Brauer–Severi scheme corresponding to $\mathcal{A}$, we have $P_H = \mathbb{P}^{r-1}_H$. Let A be the semilocal ring of the curve H corresponding to the closed points $D \subset H$. The section $\mathbb{P}(\mathcal{O}_D) \subset P_D$ is given by a surjection $A^{\oplus r} \to A/I$, where $I \subset A$ is the ideal of D. By Nakayama's Lemma, this lifts to a surjection $A^{\oplus r} \to A$. So, if we shrink X further, we can extend the section $\mathbb{P}(\mathcal{O}_D) \subset P_D$ to a section $S \subset P_H$.

Let $h : \hat{P} \to P$ be the blowing-up with center $S \subset P$, and let $E \subset \hat{P}$ be the strict transform of the Cartier divisor $P_H \subset P$. I claim that there is a birational contraction $\hat{P} \to P'$ contracting precisely the fibers of $E \to H$ to points such that P' is a Brauer–Severi scheme. Over suitable étale neighborhoods, this follows from [22] Theorem 1.4. You easily check that these contractions glue together and define a contraction in the category of schemes.

By construction, the new Brauer–Severi scheme $P' \to X$ has a trivial restriction $P'_R = \mathbb{P}^{r-1}_R$. If $\mathcal{A}'$ is the Azumaya algebra corresponding to the Brauer–Severi scheme P', this implies $\mathcal{A}'_R = \mathcal{E}nd(\mathcal{O}_R^{\oplus r})$. By the choice of the curve $R \subset Y$, this forces the formal vector bundle $\mathcal{A}'_{\mathfrak{Y}}$ to be free. Consequently, the direct image $f_*(\mathcal{A}')$ is an Azumaya $\mathcal{O}_X$-algebra representing the class $\alpha \in \mathrm{Br}'(X)$ generically. $\square$

Remark 3.2. The proof works for separated geometrically normal 2-dimensional algebraic spaces as well. This is because their resolutions are schemes.

Question 3.3. The hypothesis of *geometric* normality annoys me. What happens for separated normal 2-dimensional noetherian schemes that are of finite type over nonperfect fields, or over Dedekind rings, or have no base ring at all?

4. Existence of vector bundles

Given a scheme X, one might ask whether X admits a nonfree vector bundle. In dimension two, we can use Brauer groups to obtain a positive answer:

Theorem 4.1. *Let X be a proper normal surface over a field k. Then there is a locally free $\mathcal{O}_X$-module of finite rank that is not free.*

Proof. Seeking a contradiction, we assume that each vector bundle is free. We may assume that $k = \Gamma(X, \mathcal{O}_X)$. First, I reduce to the case that the ground field k is separably closed. Let $k \subset L$ be a separable closure, and let $\mathcal{E}_L$ be a vector bundle on $X_L = X \otimes L$, say of rank $r \geq 0$. Then there is a finite separable field extension $k \subset K$, say of degree $d \geq 1$, such that $\mathcal{E}_L$ comes from a vector bundle $\mathcal{E}_K$ on X_K. Let $p : X_K \to X$ be the canonical projection. Then $\mathcal{F} = p_*(\mathcal{E}_K)$ is a vector bundle of rank dr, hence free by assumption. This gives

$$\Gamma(X_K, \mathcal{E}_K) = \Gamma(X, \mathcal{F}) \simeq k^{\oplus rd} \simeq K^{\oplus r}.$$

Now you easily choose r sections of $\mathcal{F}$ that are linearly independent over K, which gives the desired trivialization of $\mathcal{E}_K$.

From now on, assume that k is separably closed. Note that $\operatorname{Pic}(X) = 0$, so X is nonprojective, hence it must contain some singularities. Let $f : Y \to X$ be a resolution of singularities. Choose an exceptional divisor $R \subset X$ so that $\operatorname{Pic}(R) = \operatorname{Pic}(\mathfrak{Y})$, where $\mathfrak{Y} \subset Y$ is the formal completion along the exceptional curve. The spectral sequence for $\mathbb{G}_{m,X} = f_*(\mathbb{G}_{m,Y})$ gives an exact sequence

$$0 \longrightarrow \operatorname{Pic}(Y) \longrightarrow \operatorname{Pic}(R) \longrightarrow H^2(X, \mathbb{G}_m) \to H^2(Y, \mathbb{G}_m).$$

Set $G = \operatorname{Pic}^0(R)/\operatorname{Pic}^0(Y)$, and let $H \subset \operatorname{NS}(Y)$ be the kernel of the restriction map $\operatorname{NS}(Y) \to \operatorname{NS}(R)$ for the Néron–Severi groups. The snake lemma gives an inclusion $G/H \subset H^2(X, \mathbb{G}_m)$. To proceed, we need a well-known fact:

Lemma 4.2. *For each $l > 0$ prime to $\operatorname{char}(k)$, the group $\operatorname{Pic}^0(R)$ is l-divisible.*

Proof. We have an exact sequence

$$0 \longrightarrow \operatorname{Pic}^0(R) \longrightarrow \operatorname{Pic}^0_{R/k}(k) \longrightarrow \operatorname{Br}(k).$$

Since $\operatorname{Br}(k) = 0$, we have to see that the multiplication morphism $l : P \to P$ is surjective on the smooth algebraic group scheme $P = \operatorname{Pic}^0_{R/k}$. Since P is connected, it suffices to check that $l : P \to P$ is open. The completion at the origin $0 \in P$ is a formal group, given by a formal power series ring $k[[X_1, \ldots, X_n]]$ together with n formal power series

$$F_i(X_1, \ldots, X_n, Y_1, \ldots, Y_n) = X_i + Y_i + \text{terms of higher order}.$$

Multiplication by l is given by $l-1$ substitutions

$$[l]^* X_i = F_i([l-1]^* X_1, \ldots, [l-1]^* X_n, X_1, \ldots, X_n) \equiv l X_i,$$

modulo terms of higher order. Since l is prime to the characteristic of the ground field, this is bijective. Consequently, l is étale on $\mathcal{O}^\wedge_{P,0}$, hence étale on P, and therefore open. ☐

We continue with the proof of the Proposition. If $G = 0$, then X would have nontrivial line bundles ([26], Prop. 4.2), which is absurd. So G is a nonzero l-divisible group. On the other hand, H is finitely generated. We conclude that $G/H \subset H^2(X, \mathbb{G}_m)$ contains many torsion points. Consequently, $\operatorname{Br}'(X)$ contains nonzero generically trivial classes. By Proposition 2.1 there is a nontrivial Azumaya $\mathcal{O}_X$-algebra $\mathcal{A}$.

Setting $r^2 = \operatorname{rank}(\mathcal{A})$, we have $\mathcal{A} \simeq \mathcal{O}_X^{\oplus r^2}$ as $\mathcal{O}_X$-module. The multiplication map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and the unit $\mathcal{O}_X \to \mathcal{A}$ induce a k-algebra structure on $A = \Gamma(X, \mathcal{A})$. You easily check that $A \otimes \kappa(x) \simeq \mathcal{A}(x)$ for each point $x \in X$. So A is a central simple k-algebra, which is trivial because k is separably closed. Consequently $\mathcal{A} = A \otimes \mathcal{O}_X$ is also trivial, contradiction. ☐

Remark 4.3. It might easily happen that X has trivial Picard group [25]. However, the preceding results ensures the existence of vector bundles of higher rank.

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Mathematische Fakultät, Ruhr-Universität, 44780 Bochum, Germany
Current address: M.I.T. Mathematical Department, Room 2-155, 77 Massachusetts Avenue, Cambridge, MA 02139-4307, USA
E-mail address: s.schroeer@ruhr-uni-bochum.de